Путем варьирования действия (4) по полевой функции Φ получим

$$\sqrt{-g}\frac{\partial}{\partial x^k}\left[\frac{\lambda^{jk}}{\left(c^2+\Phi\right)^2}\frac{\partial\Phi}{\partial x^j}\sqrt{-g}\right]+\frac{\lambda^{jk}}{\left(c^2+\Phi\right)^3}\frac{\partial\Phi}{\partial x^j}\frac{\partial\Phi}{\partial x^k}=-\frac{\rho}{\gamma} \tag{5}$$

(здесь $\gamma=\left(1-\frac{V^2}{c^2}\right)^{-1/2}$, ρ - плотность массы частицы).

В пределе слабого поля для неподвижного источника и при $\lambda^{jk}=\frac{c^4}{4\pi G}\eta^{jk}$ выражение (15) сводится к уравнению Пуассона. Поэтому (5) можно интерпретировать как релятивистское уравнение гравитационного поля. Кроме того, здесь Φ является скалярной функцией и должна иметь "абсолютный смысл". Материальные функции гравитирующей системы определяются как компоненты тензора энергии-импульса

$$T^k{}_l=\Phi_{,l}\frac{\partial\mathscr{L}}{\partial\Phi_{,k}}-\mathscr{L}\delta^k{}_l \tag{6}$$

где $\mathscr{L}=\lambda^{ik}E_iE_k$.

FIELD THEORY OF GRAVITATION

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The proposed field theory of gravitation, built on the analogy with electrodynamics.

Keywords: The equations of the gravitational field, classical limit, Gravitational waves.

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SOME REMARKS TO SPHERICALLY SYMMETRIC SOLUTION IN RELATIVISTIC THEORIES

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This paper discusses the split of static spherically symmetric tensors $g^{\mu\nu}$ into physical and geometrical terms in the language of "gauge" transformation. We consider the vacuum static spherically symmetric solutions of general relativity to illustrate this. This construction allowed us to reformulate significant aspects of observer dependence of physical quantities from "gauge" transformation.

Keywords: spherically symmetric solution, relativistic theories.

1. Introduction

In order to establish a link to experiments in classical theories one starts with a point set and introduces structure on them. The introduced structures can be divided into two classes, the absolute objects and the dynamical objects. Absolute object determines the behavior of dynamical objects but is not affected by these objects in turn.

Examples of absolute objects are the Minkowski space-time of special relativity, the absolute time and Euclidean space of Newtonian mechanics. Theorists have in recent years revived the old idea, going back to Immanuel Kant [9], that space and time are not absolute objects, but have to be considered as preconditions for the possibility of all experience.

This approach for convenience, one can separated in four Levels, each of which based upon following sequence: representing the minimal set of necessary component assumptions that go into the underlying space-time models of almost all physical theories. The traditional representation of physics based on the assumption that ***there is the space-time*** described as a fixed scaffolding with which to build the dynamical objects.

Level 1: Historically the quantification of geometry led to the identification of point set with Euclidean space, as absolute object, whose points are the ordered sets of n real numbers. Initially in relativity theory there is not a space-time (***which emerged only after Einstein field equations solved***) as a result a set of events exist in abstract pre-geometric set, which have not an absolute objects. Then choice of the mentioned quadruple of numbers for each points is completely arbitrary not limited by anything.

Level 2: In modern usage, a topology can be defined in terms of open sets, or their generators, neighborhoods. This leads to the notion of a topological manifold as a point set with topology which is locally Euclidean. The notation of locality is provided by the topological structure of manifold, which specifies which subset of events are open sets (neighborhoods). In relativity theory we would have no idea of which events neighbor on each others. This information is pre-determined every time by the introduced constraints as coordinate condition or "gauge" structure on set, which choice is arbitrary. Thus different "gauge" define a notation of locality have to represent different topological structure.

A pre-geometric set of relativity theory includes no concept of length or distance and no a connection - structure that allows tangent vectors of different points to be compared or related to each other. Moreover, for a pre-geometric set there is no general prescription that every point has no open neighborhood of homeomorphism to open subset R^4, so that an a priori selection criterions must be adopted.

Level 3: Current physics based on the tacit assumption that there is some natural, standard smoothness structure on that transition to third level, is trivial, a topological manifold, given by the coordinates x^μ. In all dimensions, the Euclidean space R^n with $n \neq 4$ admits a unique smoothness structure, up to diffeomorphisms. However, a four dimensional manifold have an infinity of possible smoothness structures non diffeomorphic to each other [1]. In this way relativity theory model requires the specification of its smoothness structure. This means that one can try to solve the Einstein equations on one of this non-diffeomorphic R^4. We see that Einstein gravity is quite nontrivial even in the absence of matter.

Level 4: Any physics needs the establishment of a space-time geometry by attributing its metric, connection, etc. The geometric aspects of space-time one can obtain in terms of the metric field $g^{\mu\nu}(x)$, the most important of these being, by far, the invariant interval

$$ds^2 = g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu. \tag{1}$$

These intervals are obtained on the basis of the inner product of vectors on the tangent space at every point P, it constitutes the most fundamental relationship between space-time points that remains unchanged for any coordinate system.

An unsatisfactory feature of relativity theory is that the field equations do not have any direct physical and geometrical interpretation. Relativistic theories is to choose the "gauge" on unphysical grounds, by some convention to simplify the solution of the differential equations. Unless solved the Einstein equations, we can not interpreted $g^{\mu\nu}$ geometrically; it is a parametrization of the gravitational field and nothing else in general [13]. Specifying only interval ds^2 between infinitesimally close points, the geometry of space-time is not fixed. In order to create real gravitational field one must solve the inverse task, which in general have not unique solution. If instead the metric tensor is known there are many compatible space-times. A trivial example is in the geometry of two-dimensional flat spaces, where we don't get to differentiate between a plan, a cone and a cylinder from the metric tensor. Likewise, one can use approach to gravity in which our curved space-time is considered as a surface in flat ambient space in higher dimension [3, 14]. It is know that exist six kinds of embedding with different topology of the spherically symmetric metric

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(r)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2}$$

in a flat six-dimensional space if the embedding have the symmetry of Schwarzschild solution [12].

2. Connection decomposition.

The standard treatment of general relativity through the matter action is independent from the connection that the covariant derivatives contained in the Lagrangian density of the matter field are the ones built from the metric connection. In the general relativity formulation the single object $g^{\mu\nu}$ determines at the same time, the causal structure, the length and distance and free fall of test particles. The connection constituted by coefficients with no dynamics.

The leading axiom of general relativity and therefore the only one that is usually mentioned, is the set of field equations [6]. The formal background of Einstein theory consists essentially in manifold geometry, in particular, in the theory of Riemann spaces. This is reflected by the dynamics of gravity, which is governed by the Einstein-Hilbert action [6]

$$S \;=\; \int \sqrt{-g}R(g^{\mu\nu}, \partial_\lambda g^{\mu\nu})d^4x \tag{3}$$

which together with the matter action yields the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T^M_{\mu\nu} \tag{4}$$

where Ricci tensor and scalar curvature are derived as $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$ and $R = R^\mu_\mu$. The curvature tensor associated with the connection $\Gamma^\gamma{}_{\alpha\beta}$ is defined by

$$R_{\mu\nu\rho}{}^\sigma = \partial_\nu\Gamma^\sigma{}_{\mu\rho} - \partial_\mu\Gamma^\sigma{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho}\Gamma^\sigma{}_{\nu\alpha} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\sigma{}_{\mu\alpha}\,. \tag{5}$$

Note Einstein recognized the possibility to describe the theory of general relativity assuming the independence of the affine connection from the metric [5]. It would seem that the peculiar characteristic of general relativity the possibility of a direct coupling of matter with a connection. Therefore it is quite easy to show [8, 15] that the connection can be decomposed as

$$\Gamma^{\gamma}{}_{\alpha\beta} = \left\{{\gamma \atop \alpha\beta}\right\} + \frac{1}{2}\left(-Q_{\alpha}{}^{\gamma}{}_{\beta} + Q^{\gamma}{}_{\beta\alpha} - Q_{\beta\alpha}{}^{\gamma}\right) + S_{\alpha\beta}{}^{\gamma} - S_{\beta}{}^{\gamma}{}_{\alpha} + S^{\gamma}{}_{\alpha\beta}, \tag{6}$$

where $\left\{{\gamma \atop \alpha\beta}\right\}$ is the Levi-Civita connection of the metric (also known as Christoffel symbols of the metric) and we have defined the nonmetricity tensor $Q_{\alpha\beta\gamma} \equiv \nabla_{\alpha} g_{\beta\gamma}$ and the antisymmetric part of the connection, otherwise known as the Cartan torsion tensor, $S_{\alpha\beta}{}^{\gamma} \equiv \Gamma^{\gamma}{}_{[\alpha\beta]}$.

General relativity model requires the specification of the source of the metric field through another "energy-momentum" tensor field. The relation between both tensor field is given by field equation. Furthermore, the components of metric tensor appears on both left and right sides of Einstein equations, so the manifold structure and the "matter" sources of manifold constitute a dynamical system, the equations of which can only be solved together. This circumstance raises an ambiguity in the definition of the stress-energy tensor.

However, one could equally well represent solutions of Einstein equations in any other unholonom frame so long was willing the propose a decomposition of defining stress-energy tensor as a function of nonmetricity and Cartan torsion tensors [8]. In this way the components of nonmetricity and Cartan torsion tensors in (6) transform the vacuum Einstein equation in holonom frame to field equation with matter source in unholonom frame and vice versa.

Non-linearity and incompleteness could explain the specific interplay of metric and stress-energy tensors in the right hand side of the Einstein equations and justify a particular coupling prescription for matter. The explicit form of the Schwarzschild interior

$$g^{\mu\nu} = \begin{pmatrix} \frac{3}{3-8\pi r^2} & 0 & 0 & 0 \\ 0 & r^2\mathrm{Sin}[\theta]^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & -\frac{1}{9}\left(3+\sqrt{9-24\pi r^2}\right)^2 \end{pmatrix}$$

and vacuum solutions

$$g^{\mu\nu} = \begin{pmatrix} \left(1-\frac{1}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & r^2\mathrm{Sin}[\theta]^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & -\left(1-\frac{1}{r}\right) \end{pmatrix}$$

can be of use for studying the relation between holonom and unholonom representation frame to field equation with or without matter source. Obviously, one can decomposed the Levi-Civita connection as (6) then the Cartan torsion tensor for unholonomic frames is defined by

$$
\begin{aligned}
\Psi^1_{11} &= \frac{1}{2(-1+r)r}+\frac{8\pi r}{3-8\pi r^2},\\
\Psi^1_{22} &= \frac{1}{3}\left(-3+8\pi r^3\right)\mathrm{Sin}[\theta]^2,\\
\Psi^1_{33} &= -1-2r+\frac{8\pi r^3}{3},\\
\Psi^1_{44} &= -\frac{-1+r}{2r^3}+\frac{8}{9}\pi r\left(-3+8\pi r^2-\sqrt{9-24\pi r^2}\right),\\
\Psi^4_{14} &= \frac{1}{2r-2r^2}-\frac{8\pi r}{\left(3-8\pi r^2+\sqrt{9-24\pi r^2}\right)},
\end{aligned}
\tag{7}
$$

where $\Psi^{\gamma}_{\alpha\beta}=S_{\alpha\beta}{}^{\gamma}-S_{\beta}{}^{\gamma}{}_{\alpha}+S^{\gamma}{}_{\alpha\beta}$.

In this way the we can assume the Schwarzschild interior and vacuum solutions as a same object in holonom or unholonom frame.

3. “Gauge” decomposition.

In the following, we intend to illustrate the “gauge” freedom in the context of static, spherically symmetric simple example. In the specific case of static spherically symmetric source free configurations solution of the Einstein equations is given by [10]

$$
ds^2=-\left(1-\frac{2\mu}{\sqrt{\rho(r)}}\right)dt^2+\left(\frac{\rho'(r)^2}{4\rho(r)}-\chi(r)^2\right)\left(1-\frac{2\mu}{\sqrt{\rho(r)}}\right)^{-1}dr^2+
\chi(r)drdt+\rho(r)\left(d\theta^2+\sin^2\theta d\phi^2\right),
\tag{8}
$$

where $\rho(r),\chi(r)$ are a arbitrary function of r [1]. While the description of this functions may seem foreboding they meaning can not be more geometrically clear. Thus observability is not an intrinsic property of physical object but depends also on the means of these functions. A rapid inspection of solution (8) show that the form of the metric generated by the suitable choice of functions $\rho(r),\chi(r)$ are in particular identical to solution in Hilbert or isotropic “gauge” .

In a general curved space-time, there is no way of separating part of $g^{\mu\nu}$ due to the choice of the frame/observer and part of $g^{\mu\nu}$ due to “genuine” gravitational effects [11]. Having separated the metric tensor one can use Einstein equation to define the terms of new energy momentum tensor which can be assigned to the “emergence” of a matter density and pressure. Let us consider two sets of “gauges” fixing standard Hilbert one (2) for which Einstein equations are

$$
\begin{aligned}
\frac{1}{r}+\frac{B'}{B}-\frac{A}{r}&=0,\\
\frac{r}{2A}\left(\frac{B'}{B}-\frac{A'}{A}\right)-\frac{r^2B'}{4AB}\left(\frac{A'}{A}+\frac{B'}{B}\right)+\frac{r^2B''}{2AB}&=0,\\
B\left(-\frac{1}{r}+\frac{A}{r}+\frac{A'}{A}\right)&=0,
\end{aligned}
\tag{9}
$$

1 The notation prime denote derivation with respect to r are used throughout the paper.

and the other isotropic one

$$ds^2 = -b(r)dt^2 + a(r)\left(dr^2 + r^2(r)d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{10}$$

with

$$\begin{aligned} \frac{4a'}{r} + \frac{a'^2}{a} + \frac{4ab'}{rb} + \frac{2a'b'}{b} = 0, \\ 2rba' - \frac{2r^2ba'^2}{a} + 2rab' - \frac{r^2ab'^2}{b} + 2r^2ba'' + 2r^2ab'' = 0, \\ b\left(\frac{8a'}{r} - \frac{3a'^2}{a} + 4a''\right) = 0. \end{aligned} \tag{11}$$

Suppose that, in isotropic "gauge", metric tensor (10) in (11) is decomposed into the sum

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + \widehat{g}_{\mu\nu} \tag{12}$$

of a Hilbert term and additional "physical" term

$$g_{00} \rightarrow \widehat{b} + \beta, \tag{13}$$

$$g_{11} \rightarrow \widehat{a} + \alpha, \tag{14}$$

$$g_{22} \rightarrow r^2 + \gamma, \tag{15}$$

where α, β and γ is a "physical" term. A simplifying assumption about the "physical" terms are that $\alpha = 0, \beta = 0$ and $\gamma = r^2(\widehat{a} - 1)$. Then from (11) one can obtain

$$\begin{aligned} \frac{1}{r} + \frac{\widehat{b}'}{r\widehat{b}} - \frac{\widehat{a}}{r^2} = T_{11}, \\ \frac{r}{2\widehat{a}}\left(\frac{\widehat{b}'}{\widehat{b}} - \frac{\widehat{a}'}{\widehat{a}}\right) - \frac{r^2\widehat{b}'}{4\widehat{a}\widehat{b}}\left(\frac{\widehat{a}'}{\widehat{a}} + \frac{\widehat{b}'}{\widehat{b}}\right) + \frac{r^2\widehat{b}''}{2\widehat{a}\widehat{b}} = T_{22}, \\ \frac{\widehat{b}}{r}\left(-\frac{1}{r} + \frac{\widehat{a}}{r} + \frac{\widehat{a}'}{\widehat{a}}\right) = T_{44}. \end{aligned} \tag{16}$$

For our model for $T_{\mu\nu}$, we can interpret that of an anisotropic perfect fluid with radial pressure P, tangential pressure τ and density ρ, the only nonzero elements of $T_{\mu\nu}$ are

$$T_{00} = -\rho g_{00}, T_{11} = Pg_{11}, T_{22} = \tau g_{22}, T_{33} = \tau g_{33}. \tag{17}$$

After bit of algebra one can obtain from (8) the well known solution for isotropic "gauge"

$$ds^2 = -\left(\frac{1-\mu/r}{1+\mu/r}\right)^2 dt^2 + \left(1 + \frac{\mu}{r}\right)^4 [dr^2 + r^2 d\Omega^2], \tag{18}$$

having decomposed (13) - (15) and substituting this in the (16) we obtain

$$P = \frac{5r^2\mu^4 + 4r\mu^5 + \mu^6 - 6r^4\mu^2}{r^2(r-\mu)(r+\mu)^5}, \tag{19}$$

$$\tau = \frac{2r^2\mu}{(r-\mu)}\frac{\left(2r^2 - 3r\mu - \mu^2\right)}{(r+\mu)^6}, \tag{20}$$

$$\rho = \frac{1}{r^2} - \frac{r^2(r+5\mu)}{(r+\mu)^5} \tag{21}$$

The solution that we obtain show the anticipated property that the stress energy tensor do not vanish, i.e. the “gauge” decomposition ‘creates” a matter.

An intriguing consequences of the above proposition in relativistic theories is the occurrence of the researcher by “gauge” fixing influences results of measurements and physics and geometry are different in a different “gauges”.

4. Discussion

We have shown that in general relativity a “gauge” transformation leads to a certain transformation of all physical and geometrical quantities and, as consequences “gauge” and coordinate transformation are two different manipulations. This is because the Einstein field equations is written in pregeometric manifold, while observations are usually analyzed in the physical space-time endowed with its Riemannian metric. The parameters x^μ on an pregeometric manifold M do not identify an operationally well-defined position in space-time, although they can be regarded as defining a “gauge”. While the points of the manifold have an inherent essence as elements of space-time, they lack uniqueness as individualized points of that space-time (events) unless and until a explicit metric tensor and non-geometrical fields are specified.

A coordinate system give us a numerical labelling, which enables us to distinguish points of space-time from one another, *which can be obtained only after Einstein field equations solved.* Thus, coordinates of space-time in general relativity are physically meaningless before specifying the metric tensor though they designate a particular point of underlying manifold. In order to avoid this ambiguity, the values of four independent invariants of the metric fields are supposed to individuate the space-time points in the generic case.

In the realm of general relativity “gauge” transformation may be interpreted as decomposition of metric tensor into the sum of “physical” and “geometrical” terms associated with coordinate choice [2]. The related transformation of stress-energy tensor of Einstein equations can be obtain directly by moving the all emerges “non-geometric” terms into right hand side. The physical property of gravitational field that we obtain show the anticipated property that the deviation of stress energy tensor do not vanish, i.e. the “gauge” transformation “creates” a matter.

Indeed approaches, that attempt to describe universe as a solution of Einstein’s equations add more sophistication to the picture, as even concepts of space-time is not well defined and, as consequences the notation of holonom and unholonom frame has no clear interpretation. It is easy to show, for example, that the Schwarzschild interior and vacuum solutions one can interpreted as a same object in holonom or unholonom frame.

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